

On a dynamic adaptation of the Distribution Builder approach to investment decisions

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First draft: June 2011

This draft: November 2012

Abstract

Sharpe *et al.* proposed the idea of having an expected utility maximizer choose a probability distribution for future wealth as an input to her investment problem instead of a utility function. They developed a computer program, called *The Distribution Builder*, as one way to elicit such a distribution. In a single-period model, they then showed how this desired distribution for terminal wealth can be used to infer the investor's risk preferences. We adapt their idea, namely that a risk-averse investor can choose a desired distribution for future wealth as an alternative input attribute for investment decisions, to continuous time. In a variety of scenarios, we show how the investor's desired distribution combines with her initial wealth and market-related input to determine the feasibility of her distribution, her implied risk preferences, and her optimal policies throughout her investment horizon. We then provide several examples.

Keywords: inferring preferences, Distribution Builder, expected utility, forward investment performance

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1 Introduction

Theoretical models in single-agent investment are traditionally based on the classical criterion of maximal expected utility of wealth. Despite its long history and sound economic foundations, however, expected utility as a criterion for practical investment choice faces many obstacles due to various difficulties for its specification. Some of these difficulties have been addressed by making simplifying or ad hoc assumptions. Asset managers, for instance, often make two such assumptions. First, they assume that the investor has constant relative risk aversion. They then use so-called risk tolerance quizzes to approximate the investor's relative risk aversion coefficient.

Alternatively, one can focus on observable features of investors' behavior. For instance, Black (1988), among others, proposed to essentially bypass the utility concept altogether and, instead, use the investor's initial choice of optimal investment as the criterion to determine future optimal allocations. In a related direction, several papers have studied the specification of utility if one knows *a priori* the optimal allocations that are consistent with this utility (see, among others, Cox and Leland (2000), He and Huang (1994), Dybvig and Rogers (1997) and Cox *et al.* (2011)).

Sharpe and his collaborators took a different point of view in Sharpe *et al.* (2000), Sharpe (2001), and Goldstein *et al.* (2008). They argued that, in practical situations, investors can express desires about the distribution of their future wealth. To gather such distributional data, they developed a computer program, called *The Distribution Builder*, whose output is a probability distribution that the investor desires for her future wealth. Then, in a single-period model and under the assumption that the investor implicitly maximizes her expected utility of terminal wealth, Sharpe *et al.* showed how this desired distribution can be used to recover the investor's risk preferences.

Our work is inspired and motivated by this approach. The aim herein is to provide a dynamic adaptation of their idea, which is to use a risk-averse investor's desired distribution for future wealth, rather than a utility function, as an input for optimal investment. Given an investor's desired distribution for future wealth and her initial endowment, we study the following issues: if this distribution can be achieved in the market, how it is achieved, and, finally, the risk preferences that are consistent with

this choice of distribution. As in the work of Sharpe *et al.*, we address, in a practical way, both the normative issue of instructing investors how to achieve their goals as well as the theoretical question of how to infer risk preferences that are consistent with investment targets.

Given that we work beyond a single-period setting, the time at which the investor wants to achieve her desired distribution is an important input parameter in the analysis. We consider two scenarios. In the first, we assume that the investor implicitly maximizes her expected utility of terminal wealth in a *fixed horizon setting*, by which we mean that the investor has a finite and fixed investment horizon that is specified when investment begins. Within the fixed horizon setting, we consider two subcases depending on whether the investor targets her distribution for terminal wealth or for wealth at some intermediate time. This scenario is appropriate for an investor who is certain about the length of her investment horizon and is not interested in exploring investment opportunities beyond it while she is investing. In the second scenario, we assume that the investor operates in a *flexible horizon setting*, by which we mean that the time at which investment ends is not predetermined and could be finite or infinite. The investor places her chosen distribution for wealth at some arbitrary future time. This scenario is appropriate for an investor who does not want to commit at initial time to a fixed investment horizon, or plans to invest for a very long time.

The market environment that we consider consists of risky stocks and a riskless money market account. The stock prices are modeled as geometric Brownian motions with time-varying deterministic coefficients.

Our results are as follows. In the fixed horizon setting, we show that the desired distribution, the investor's initial wealth, and market-related input are sufficient to explicitly determine the feasibility of the investor's choice of distribution, the optimal strategy the investor should follow to attain her goal, and the investor's terminal marginal utility function. We obtain these results regardless of whether the investor targets her distribution for terminal wealth or for wealth at an intermediate time.

We obtain analogous results for the flexible horizon setting. Here, the terminal-horizon expected maximal utility criterion needs to be modified, and for this we use the

so-called monotone forward investment performance criterion. Again, we show that the investor's desired distribution, her initial wealth, and market-related input are sufficient to determine the feasibility of the distribution, the strategy that achieves it, and her risk preferences.

In the fixed horizon setting, the method of proof relies on known representation results for the optimal wealth process in terms of the solution to the heat equation and on the work of Widder on inverting the Weierstrass transform. In the flexible horizon setting, it is shown that the investor's distribution, initial wealth, and market input determine the Fourier transform of a particular Borel measure that is known to characterize all objects of interest in the model under the monotone forward investment performance investment criterion.

Our results show that in our model, a desired distribution for wealth at a *single* future time, when combined with the investor's initial wealth and an estimate of the market price of risk throughout the investment horizon, explicitly determines the investor's risk preferences, her optimal policies throughout, and the feasibility of her chosen distribution. This result holds regardless of whether the investor is a classical expected utility maximizer with a fixed investment horizon or whether she uses the monotone forward investment performance criterion with a flexible investment horizon.

The paper is organized as follows. In section 2, we review the method underlying *The Distribution Builder*. In section 3, we present the continuous-time model and relevant background results on the expected utility and monotone forward investment performance investment criteria. In section 4, we consider targeted wealth distributions in the fixed horizon setting, while in section 5 we consider targeted wealth distributions in the flexible horizon setting. We provide conclusions and directions for future research in section 6.

2 Single-period investment model and its Distribution Builder

To motivate the reader, we review the model setting and the method of *The Distribution Builder* developed by Sharpe *et al.* (see Sharpe *et al.* (2000), Sharpe (2001), and Goldstein *et al.* (2008)). Therein, three key model assumptions were made: i) the state price density is solely expressed in terms of the stock price, ii) the investor is implicitly an expected utility maximizer, but specifies her desired future wealth distribution instead of her utility function, and iii) the investor wants to obtain her desired distribution in a so-called cost-efficient manner. We elaborate on their model and on these assumptions next.

The model is a single-period one having $N > 2$ distinct possible states $\Omega := \{\omega_i\}_{i=1}^N$, each occurring with equal probability $\mathbb{P}\{\omega_i\} = \frac{1}{N}$, $i = 1, \dots, N$. The market consists of one riskless money market and one risky stock. The former has initial price $B_0 = 1$ and is assumed to offer constant interest rate $r > 0$, i.e. $B_T(\omega_i) = (1+r)$, $i = 1, \dots, N$.

The stock has initial price $S_0 = 1$ and its terminal values in the N states are determined by a discrete approximation to a lognormal distribution. This is accomplished as follows. The logarithmic return of the stock is assumed to be normally distributed with mean $\mu > 0$ and standard deviation $\sigma > 0$. The resulting continuous distribution is then lognormal and can be approximated by selecting N points with probabilities $\frac{1}{2N}, \frac{3}{2N}, \dots, \frac{2N-1}{2N}$ from the inverse of its cumulative distribution function. This in turn produces the vector S_T of N equally probable states. Without loss of generality, it is assumed that the states are in nondecreasing order,

$$S_T(\omega_i) \leq S_T(\omega_{i+1}), \quad i = 1, \dots, N-1. \quad (1)$$

Moreover, to preclude arbitrage in this model, the familiar assumption $S_T(\omega_1) < 1+r < S_T(\omega_N)$ is introduced.

The market admits a state price density vector ξ_T , which is not unique because of incompleteness. Sharpe *et al.* then make the ad hoc assumption that the logarithm of

the vector ξ_T satisfies the linear relationship

$$\log(\xi_T(\omega_i)) = a + b \log(S_T(\omega_i)), \quad i = 1, \dots, N, \quad (2)$$

for some constants a and b . To find these constants, one uses the identities

$$\frac{1}{N} \sum_{i=1}^N \xi_T(\omega_i) = \frac{1}{1+r} \quad \text{and} \quad \frac{1}{N} \sum_{i=1}^N \xi_T(\omega_i) S_T(\omega_i) = S_0 = 1, \quad i = 1, \dots, N,$$

to derive the equation

$$(1+r) \sum_{i=1}^N S_T^b(\omega_i) = \sum_{i=1}^N S_T^{b+1}(\omega_i). \quad (3)$$

This equation then determines b and using (2) we, in turn, find a . It is easily shown that if $\mu > r$ then the solution b to (3) exists, is unique, and is strictly negative.

The assumption that the stock price and state price density are related as in (2) seems at first to be restrictive and arbitrary. This relationship, however, is consistent with widely used models of asset prices, examples of which include multiperiod iid binomial models in discrete time and the classical Black-Scholes-Merton model in continuous time (see Sharpe (2001) for further discussion).

In this market environment, the investor starts with initial wealth $x_0 > 0$ and sets an investment goal, namely a probability distribution denoted by F , for her terminal wealth. As we describe in detail below, the issue of whether F can be attained depends on x_0 and on market-related input. To achieve an attainable distribution, the investor chooses at initial time how much money π to allocate to the risky asset, with the remaining quantity $x_0 - \pi$ invested in the money market. Her wealth at time T is, then, given by the random variable (recall $S_0 = 1$)

$$X_T(\omega) = \pi S_T(\omega) + (x_0 - \pi)(1+r).$$

The wealth distribution F is characterized by its probability mass function, namely

$$\mathbb{P}(X_T = x_i) = \frac{n_i}{N}, \quad n_i \in \{0, 1, \dots, N\}, \quad i = 1, \dots, N.$$

Therefore, F can be viewed as an N -vector, $\bar{X}^F = \{x_i^F\}_{i=1}^N$, of wealth values where, for each $i = 1, \dots, N$, we assign n_i values equal to x_i . Without loss of generality, the

values of \bar{X}^F are assumed to be in nondecreasing order, i.e. $x_i^F \leq x_{i+1}^F$, $i = 1, \dots, N$. Given this assumption, there is a one-to-one correspondence between the distribution F and the wealth vector \bar{X}^F , in the sense that for every distribution F there is a given wealth vector \bar{X}^F , and vice-versa.

To find a terminal wealth random variable X_T with a given distribution F , one associates each of the N values in the vector \bar{X}^F with one of the N states of the world. There are $N!$ possible such bijections and each has a potentially different associated cost. For fixed $j = 1, \dots, N!$, let $X_T^j: \Omega \rightarrow \bar{X}^F$ be such a bijection. Then, the cost of the distribution F attained using the random variable X_T^j is found by computing the inner product $C(j)$, defined by

$$C(j) = \frac{1}{N} \sum_{i=1}^N \xi_T(\omega_i) X_T^j(\omega_i).$$

Sharpe *et al.* assume that the investor is implicitly choosing a distribution that maximizes her expected utility of terminal wealth. In a complete market, it is well known that the optimal strategy of an investor who maximizes expected terminal utility is *cost-efficient*, i.e. it achieves the so-called *distributional price*

$$P_D(F) := \min_{j=1, \dots, N!} C(j) \quad (4)$$

of the distribution F (see Dybvig (1988a) and Dybvig (1988b)). This is not true, however, in the incomplete market herein. The optimal strategy is not necessarily cost-efficient. Nevertheless, Sharpe *et al.* assume that the investor does prefer to obtain her desired distribution F using a cost-efficient strategy. One can then use the results of Dybvig (1988a) to deduce that the strategy j^* , defined by

$$X_T^{j^*}(\omega_i) = x_i^F, \quad i = 1, \dots, N, \quad (5)$$

is cost-efficient. Moreover, if j^* also satisfies $C(j^*) \leq x_0$, then it corresponds to the optimal investment strategy for the investor maximizing her expected utility of terminal wealth.

We are now ready to review the results of Sharpe *et al.* on how to infer points on the investor's marginal utility curve from her desired distribution F . Given a wealth

distribution F , one first determines the random variable X_T^{j*} via (5). Points along the marginal utility curve are then determined by the first order conditions of the investor's utility maximization problem, which are

$$U'_T(X_T^{j*}(\omega_i)) = k\xi_T(\omega_i), \quad i = 1, \dots, N, \quad (6)$$

and

$$k(C(j^*) - x_0) = 0,$$

where $k \geq 0$ is the Lagrange multiplier associated with the budget constraint $C(j) \leq x_0$. We recall that the strict positivity of the marginal utility function U'_T guarantees that $k > 0$ and, therefore, the budget constraint is binding, i.e. $C(j^*) = x_0$. Hence, it is optimal for an expected utility maximizer to select a distribution F whose distributional price in (4) is equal to her entire initial budget x_0 .

To summarize, the investor chooses a distribution F for her terminal wealth that she would like to achieve by investing her initial wealth $x_0 > 0$. It is assumed that the investor would like to achieve this distribution in a cost-efficient manner and that she implicitly maximizes the expected utility of her terminal wealth. These assumptions then determine the budget constraint that F must satisfy, namely

$$x_0 = C(j^*) = \frac{1}{N} \sum_{i=1}^N \xi_T(\omega_i) x_i^F,$$

where $\{x_i^F\}_{i=1}^N$ is the representation of F as an N -vector as described above. Furthermore, the pointwise specification of the investor's optimal terminal wealth random variable is given by (5). The investor's risk preferences are then described by an N -point approximation of the investor's marginal utility curve given by (6). Finally, the model (a one period model with N possible states) is incomplete for $N > 2$, and so it is not possible to uniquely determine the optimal initial allocation π to the risky stock.

2.1 The Distribution Builder interface: How a user selects a desired distribution for her future wealth

We briefly discuss an example using *The Distribution Builder* so that the reader will be acquainted with one possible procedure for choosing a desired distribution for future

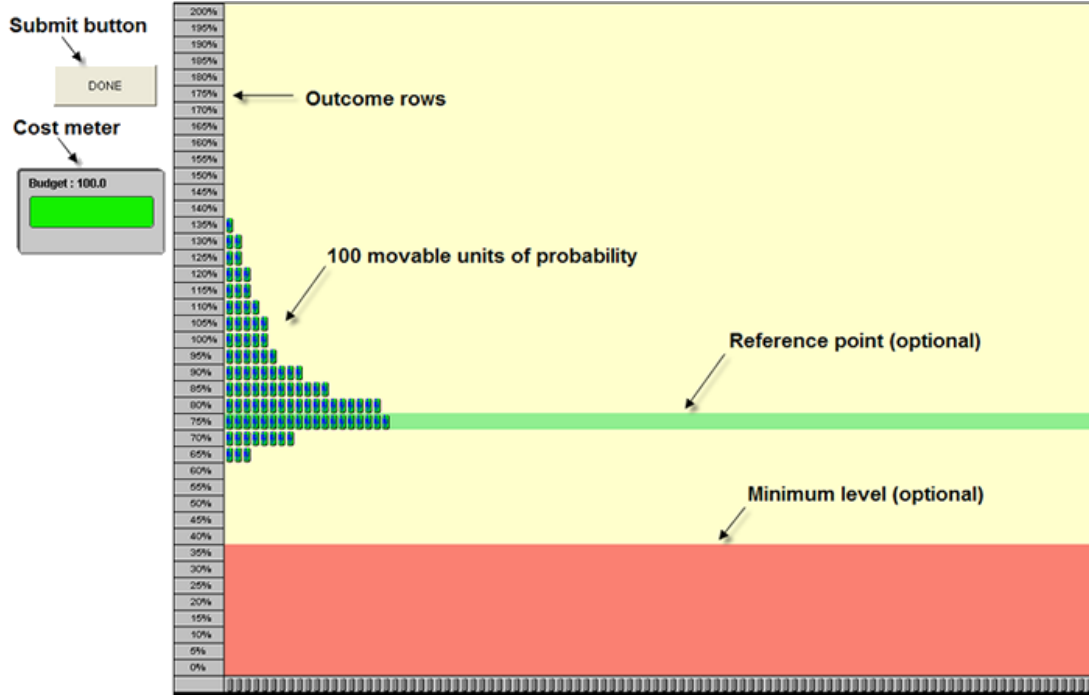


Figure 1: The Distribution Builder User Interface. Reprinted from Sharpe *et al.* (2000).

wealth. We note, however, that in our continuous-time work herein we assume that the investor chooses a distribution for future wealth, but we do not investigate specific ways or tools she might use for this purpose.

The following example comes from a specific application of *The Distribution Builder*, namely to elicit a desired probability distribution for the user's income per year following retirement. The interface for this application of *The Distribution Builder* is pictured in figure 1. The vertical axis of percentages corresponds to the percentage of pre-retirement income that will be realized annually in retirement. For example, if the investor earned \$100,000 in the year before retirement, the 75% row corresponds to a subsequent annual retirement income of \$75,000.

In an experimental setting, users are told that some reference point, which is 75% in figure 1, is a typically recommended goal for annual retirement income. The reference row can then be calibrated to represent the level of wealth that can be attained with certainty by investing in the risk-free asset.

The main area of the interface contains 100 markers, which are initially positioned along the bottom of the screen. Each marker represents an equally-likely state of the world, and the user is told that her realized outcome is represented by one of these markers. Users are only able to submit distributions of a given fixed cost (expressed as a percentage), and the cost meter on the left hand side of the interface adjusts accordingly as the user places markers along the vertical axis. The user can submit a distribution of markers only when the cost meter indicates that between 99 and 100 percent of the total fixed budget has been consumed. When satisfied with a particular distribution that meets the cost requirement, the user submits it and the computer then removes all but one of the markers, so that the user is able to experience the actual realization of her desired distribution.

3 The continuous-time model and background results on investment performance criteria

We describe the market setting in which our investor operates, as well as known results on related investment performance criteria. The background results concerning these criteria will be used in the fixed horizon setting in section 4 and the flexible horizon setting in section 5.

The market is complete and consists of a riskless money market and d risky assets driven by d independent Brownian motions. The risky assets are modeled by time-dependent geometric Brownian motions on \mathbb{R}^d , i.e. for $i = 1, \dots, d$, the price $S_t^i, t \geq 0$, of the i -th risky asset satisfies

$$dS_t^i = S_t^i \left(\mu^i(t)dt + \sum_{j=1}^d \sigma^{ji}(t)dW_t^j \right), \quad S_0^i > 0, \quad (7)$$

where $\mu^i(t)$ and $\sigma^{ji}(t)$ are deterministic functions of time for $i, j = 1, \dots, d$, and $t \geq 0$. Here, $W = (W^1, \dots, W^d)$ is a d -dimensional standard Brownian motion (regarded as a column vector) defined on a complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ where the filtration $(\mathcal{F}_t)_{t \geq 0}$ satisfies the usual conditions. It is assumed that $\mu^i(t)$ and $\sigma^{ji}(t)$ are uniformly bounded in $t \geq 0$, for all i, j . For brevity, we write $\sigma(t)$ to denote

the volatility matrix, i.e. the $d \times d$ matrix $(\sigma^{ji}(t))$ whose i -th column represents the volatility $\sigma^i(t) = (\sigma^{1i}(t), \dots, \sigma^{di}(t))$ of the i -th risky asset. We also assume that the matrix function $\sigma(t)$ is invertible for all $t \geq 0$, and we will write this inverse as $\sigma^{(-1)}(t)$. We can then alternatively write (7) as

$$dS_t^i = S_t^i(\mu^i(t) dt + \sigma^i(t) \cdot dW_t). \quad (8)$$

The riskless money market has price process $B_t, t \geq 0$, satisfying $B_0 = 1$ and

$$dB_t = r(t)B_t dt, \quad (9)$$

for a nonnegative time-dependent interest rate function $r(t), t \geq 0$, which is assumed to be uniformly bounded in $t \geq 0$. We denote by $\mu(t)$ the $d \times 1$ vector with coordinates $\mu^i(t)$ and by $\mathbf{1}$ the d -dimensional vector with every component equal to one.

We define the function $\lambda(t), t \geq 0$, by

$$\lambda(t) := (\sigma^\top(t))^{(-1)}(\mu(t) - r(t)\mathbf{1}), \quad (10)$$

and we will occasionally refer to it as the *market price of risk*.

Assumption 1. The function $\lambda(t), t \geq 0$, is continuous and uniformly bounded on $t \geq 0$. Furthermore, its Euclidean norm, $|\lambda(t)|, t \geq 0$, is Hölder continuous, and there exist positive constants c_0 and c_1 such that $0 < c_0 \leq |\lambda(t)| \leq c_1$ for all $t \geq 0$.

Starting at time $t_0 = 0$ with initial endowment $x_0 > 0$, the investor invests dynamically in the risky assets and the riskless one. The present values of the amounts invested in the assets are denoted by $\pi_t^i, i = 1, \dots, d$, and by π_t^0 , respectively. The present value of her total investment is then given by $X_t^\pi = \sum_{i=0}^d \pi_t^i$, which we will refer to as the discounted wealth generated by the (discounted) strategy $\pi = (\pi_t^0, \pi_t^1, \dots, \pi_t^d)$. The investment strategies π play the role of control processes and are assumed to be self-financing. Using (8), (9) and (10) we deduce

$$dX_t^\pi = \sigma(t)\pi_t \cdot (\lambda(t)dt + dW_t), \quad t > 0, \quad (11)$$

where $\pi_t = (\pi_t^i; i = 1, \dots, d)$ is a column vector.

The investor selects a portfolio process from an admissibility set \mathcal{A} . A detailed description of this set is given in the upcoming sections.

Finally, we introduce the auxiliary market input processes A_t and M_t , $t \geq 0$, defined by

$$A_t = \int_0^t |\lambda(s)|^2 ds \quad \text{and} \quad M_t = \int_0^t \lambda(s) \cdot dW_s. \quad (12)$$

We also recall the martingale Z_t , $t \geq 0$, given by

$$Z_t = \exp \left\{ - \int_0^t \lambda(s) \cdot dW_s - \frac{1}{2} \int_0^t |\lambda(s)|^2 ds \right\} = \exp \left\{ -M_t - \frac{1}{2} A_t \right\}. \quad (13)$$

3.1 Background results on classical expected utility theory

We briefly review background results on the classical expected utility theory. These results will be relevant in the fixed horizon setting considered in section 4.

The investor invests in $[0, T]$, with $T > 0$ being arbitrary but fixed. She derives utility only from terminal wealth, with objective

$$v(x_0, 0) := \sup_{\pi \in \mathcal{A}_T} \mathbb{E} [U_T(X_T^\pi) | X_0^\pi = x_0]. \quad (14)$$

The set of admissible policies \mathcal{A}_T is defined as the set of \mathcal{F}_t -progressively measurable and self-financing portfolio processes π_t , $t \in [0, T]$, such that $\mathbb{E} \int_0^T |\sigma(s)\pi_s|^2 ds < \infty$, and $X_t^\pi \geq 0$, $t \in [0, T]$, \mathbb{P} -a.s., where X_t^π solves (11). We will call an investor with the above investment paradigm a *Merton investor*.

The utility function $U_T(\cdot)$ satisfies the following standard assumptions.

Assumption 2. (i) The function $U_T: (0, \infty) \rightarrow \mathbb{R}$ is twice continuously differentiable, strictly increasing, and strictly concave.

(ii) The Inada conditions,

$$\lim_{x \downarrow 0} U_T'(x) = \infty \quad \text{and} \quad \lim_{x \uparrow \infty} U_T'(x) = 0, \quad (15)$$

are satisfied

(iii) The inverse, $I_T: (0, \infty) \rightarrow (0, \infty)$, of the investor's marginal utility function U_T' has polynomial growth, i.e. there is a constant $\gamma > 0$ such that

$$I_T(y) \leq \gamma + y^{-\gamma}. \quad (16)$$

The stochastic optimization problem (14) has been extensively studied and completely solved (see, for example, Karatzas and Shreve (1998)).

The following result relates the Merton investor's optimal wealth process and optimal portfolio process to the solution of the heat equation. It is well known that the optimal policies in this model can be written in terms of a solution to a linear parabolic terminal value problem (see, for example, Karatzas and Shreve (1998, Lemma 8.4 (p. 122))), but the idea of writing the optimal policies specifically in terms of the solution of the heat equation first appeared in Källblad (2011) in the lognormal setting. We state the results of Källblad (2011) next.

Proposition 3.1. *Let $x_0 > 0$ be the investor's initial wealth and let $\lambda(t)$ be as in (10). Let $h: \mathbb{R} \times [0, T] \rightarrow (0, \infty)$ be the unique solution to*

$$\begin{cases} h_t + \frac{1}{2}|\lambda(t)|^2 h_{xx} = 0, & (x, t) \in \mathbb{R} \times [0, T) \\ h(x, T) = I_T(e^{-x}), & x \in \mathbb{R}, \end{cases} \quad (17)$$

with I_T satisfying (16). Then, the following hold.

i) *The optimal wealth process $X_t^*, t \in [0, T]$, is given by*

$$X_t^* = h\left(h^{(-1)}(x_0, 0) + A_t + M_t, t\right), \quad t \in [0, T], \quad (18)$$

where A_t and M_t , $t \in [0, T]$, are defined in (12) and $h^{(-1)}$ is the spatial inverse of h .

ii) *The optimal portfolio process $\pi_t^*, t \in [0, T]$, that generates X_t^* is given by*

$$\pi_t^* = h_x\left(h^{(-1)}(X_t^*, t), t\right) \sigma^{(-1)}(t) \lambda(t), \quad t \in [0, T]. \quad (19)$$

3.2 Background results on forward investment performance processes

We now review results on the so-called forward investment performance process. These results will be relevant for the flexible investment horizon setting of section 5. The forward investment performance process is an investment selection criterion developed by Musiela and Zariphopoulou (see, among others, Musiela and Zariphopoulou (2008, 2009, 2010)) as a complementary alternative to the maximal expected utility theory.

The main motivation for this approach is the ability to work in flexible investment horizon settings and define for them time-consistent performance criteria for all times. In this framework, an admissible investment strategy is deemed optimal if it generates a wealth process whose average performance is maintained over time. In other words, the average performance of the optimal strategy at any future date, conditional on today's information, preserves the performance of this strategy up until today. Any strategy that fails to maintain the average performance over time is then suboptimal. In contrast to the expected utility criterion considered earlier, the forward investor does not specify her risk preferences for some terminal time. Instead, her risk preferences are specified at initial time by an initial datum u_0 and then evolve dynamically forward in time for $t \geq 0$.

Next, we recall the forward investment performance process. The set of admissible strategies, \mathcal{A} , is defined to be the set of \mathcal{F}_t -progressively measurable and self-financing portfolio processes $\pi_t, t \geq 0$, such that $\mathbb{E} \int_0^t |\sigma(s)\pi_s|^2 ds < \infty, t > 0$, and $X_t^\pi \geq 0, t \geq 0$, $\mathbb{P} - a.s.$, where the discounted wealth process solves (11).

Definition 1. Let $u_0: (0, \infty) \rightarrow \mathbb{R}$ be strictly concave and strictly increasing. An \mathcal{F}_t -adapted process $U(x, t)$ is a forward investment performance if, for $t \geq 0$ and $x \in (0, \infty)$:

- (i) $U(x, 0) = u_0(x)$,
- (ii) the map $x \mapsto U(x, t)$ is strictly concave and strictly increasing,
- (iii) for each $\pi \in \mathcal{A}$, $\mathbb{E}[U(X_t^\pi, t)^+] < \infty$ and $\mathbb{E}[U(X_s^\pi, s)|\mathcal{F}_t] \leq U(X_t^\pi, t)$, $s \geq t$,
- (iv) there exists $\pi^* \in \mathcal{A}$ for which $\mathbb{E}[U(X_s^{\pi^*}, s)|\mathcal{F}_t] = U(X_t^{\pi^*}, t)$, $s \geq t$.

We refer the reader to Musiela and Zariphopoulou (2009) as well as Källblad (2011) for further discussion on the forward investment performance and its similarities and differences with the classical value function.

3.2.1 Review of monotone forward investment performance processes

We focus herein on the class of time-decreasing forward investment performance processes that will be used in our analysis in section 5. These processes were introduced in Musiela and Zariphopoulou (2009) and Berrier *et al.* (2009) and further analyzed in

Musiela and Zariphopoulou (2010). Therein, it was shown that time-decreasing forward investment performance processes $U(x, t)$ are constructed by compiling market-related input with a deterministic function of space and time. Specifically, for $t \geq 0$, we have

$$U(x, t) = u(x, A_t) \quad (20)$$

where $A_t, t \geq 0$, is as in (12) and $u(x, t)$ is a smooth function that is spatially strictly increasing and strictly concave, and satisfies

$$\begin{cases} u_t - \frac{1}{2} \frac{u_x^2}{u_{xx}} = 0, & (x, t) \in (0, \infty) \times (0, \infty) \\ u(x, 0) = u_0(x), & x \in (0, \infty) \end{cases} \quad (21)$$

where $u_0: (0, \infty) \rightarrow \mathbb{R}$ is the initial datum of Definition 1.

It is also shown in Musiela and Zariphopoulou (2010) that if $h(x, t)$ is defined via the transformation

$$u_x(h(x, t), t) = e^{-x + \frac{t}{2}}, \quad (x, t) \in \mathbb{R} \times [0, \infty), \quad (22)$$

then it is a positive and spatially strictly increasing space-time harmonic function, solving the ill-posed heat equation

$$\begin{cases} h_t + \frac{1}{2} h_{xx} = 0, & (x, t) \in \mathbb{R} \times [0, \infty) \\ h(x, 0) = (u'_0)^{(-1)}(e^{-x}), & x \in \mathbb{R}. \end{cases} \quad (23)$$

Moreover, the associated optimal processes X_t^* and π_t^* , $t \geq 0$, can be written explicitly in terms of market-related input and the function h , namely, for $t \geq 0$,

$$X_t^* = h \left(h^{(-1)}(x_0, 0) + A_t + M_t, A_t \right) \quad (24)$$

and

$$\pi_t^* = h_x \left(h^{(-1)}(X_t^*, A_t), A_t \right) \sigma^{(-1)}(t) \lambda(t), \quad (25)$$

where A_t and M_t , $t \geq 0$, are as in (12) and the function $h^{(-1)}$ stands for the spatial inverse of h .

As mentioned above, problem (23) (and, in turn, (21)) are ill-posed. Nevertheless, as we review next, solutions do exist, though we expect the set of admissible initial data $u(x, 0)$ and $h(x, 0)$ to be rather restricted. We elaborate on this in Remark 3.7.

From the above, one observes that all objects of interest, including the risk preferences of the investor, her optimal strategies, and the associated forward investment performance process, are determined once the functions u and h are known and the market price of risk is chosen (which yields the processes A_t and M_t). The study of the functions u and h is therefore crucial to the understanding of the (forward) portfolio choice problem.

Remark 3.2. Recall from Proposition 3.1 that a representation of the optimal policies similar to (24) and (25) holds in the expected utility case. Note, however, that the harmonic function therein depends on market parameters while, in the monotone forward investment performance case, it does not (cf. (27)).

3.2.2 Analysis of the functions u and h

We recall some known analytical results concerning the representation of, and connections between, the functions u and h . Using Widder's classical theorem, it was shown in Musiela and Zariphopoulou (2010) that positive and spatially strictly increasing space-time harmonic functions h can be represented in terms of a Borel measure ν that has finite Laplace transform and support in the positive reals. Given such a representation, the function u is constructed using (22). Since the risk preferences and optimal strategies of the investor are represented in terms of the functions u and h (cf. (20), (24), and (25)), the measure ν emerges as the defining element in the entire analysis of monotone forward investment performance processes. We specify ν in detail next.

Define $\mathcal{B}(\mathbb{R}^+)$ to be the set of finite Borel measures ν on \mathbb{R} such that $\nu((-\infty, 0]) = 0$, and consider the following subset of $\mathcal{B}(\mathbb{R}^+)$:

$$\mathcal{B}^+(\mathbb{R}^+) = \left\{ \nu \in \mathcal{B}(\mathbb{R}^+) : \int_{0+}^{\infty} \frac{\nu(dy)}{y} < \infty \text{ and } \int_0^{\infty} e^{yx} \nu(dy) < \infty, x \in \mathbb{R} \right\}. \quad (26)$$

The following result can be found in Musiela and Zariphopoulou (2010).

Proposition 3.3. *i) Let $\nu \in \mathcal{B}^+(\mathbb{R}^+)$. Then, the function $h: \mathbb{R} \times [0, \infty) \rightarrow (0, \infty)$ defined by*

$$h(x, t) = \int_{0+}^{\infty} \frac{e^{yx - \frac{1}{2}y^2t}}{y} \nu(dy) \quad (27)$$

is a solution to (23) that is positive and spatially strictly increasing.

ii) Conversely, let $h: \mathbb{R} \times [0, \infty) \rightarrow (0, \infty)$ be a positive and spatially strictly increasing solution to (23). Then, there exists $\nu \in \mathcal{B}^+(\mathbb{R}^+)$ such that h is given by (27).

Remark 3.4. The proof of Proposition 3.3 is based on the classical result of Widder that characterizes nonnegative and spatially strictly increasing solutions to the backward heat equation on the half line $t \in [0, \infty)$ in terms of a Borel measure ν with finite Laplace transform. An analogous representation result can be obtained in the classical maximal expected utility case for solutions to the related terminal value problem (17). Indeed, one can show (see Widder (1975) and Wilcox (1980)) that h solves (17) if and only if there exists a Borel measure $\tilde{\nu}$ on \mathbb{R} such that

$$\int_{-\infty}^{\infty} e^{-\frac{y^2}{2t}} \tilde{\nu}(dy) < \infty, \quad t \in (0, T)$$

and

$$h(x, t) = \frac{1}{\sqrt{2\pi(A_T - A_t)}} \int_{-\infty}^{\infty} e^{-\frac{1}{2} \frac{(x-y)^2}{(A_T - A_t)}} \tilde{\nu}(dy), \quad (x, t) \in \mathbb{R} \times (0, T).$$

In the expected utility case, we deduce via (17) that the measure $\tilde{\nu}$ is absolutely continuous with respect to Lebesgue measure and is given by

$$\tilde{\nu}(dy) = I_T(e^{-y})dy,$$

where I_T is the inverse of the investor's marginal utility U'_T . Thus we see from Proposition 3.1 that all objects of interest in the classical expected utility model are also specified once the market price of risk and a Borel measure encapsulating the investor's preferences are chosen. A parallel result holds in the monotone forward investment performance case, as we will see below in Theorem 3.1 and Remark 3.8.

The next result characterizes analytically the set of measures $\mathcal{B}^+(\mathbb{R}^+)$ and provides a method by which one can find the measure ν given the function h . It will play a central role in the proof of Theorem 5.1.

Proposition 3.5. i) A Borel measure ν is in $\mathcal{B}^+(\mathbb{R}^+)$ if and only if its Laplace transform is entire and $\int_{0+}^{\infty} \frac{\nu(dy)}{y} < \infty$.

ii) Let h be given by (27) for some $\nu \in \mathcal{B}^+(\mathbb{R}^+)$. The mapping $x \mapsto h_x(x, 0)$ is the Laplace transform of ν and it has a unique analytic extension to \mathbb{C} . Moreover, the mapping

$$x \mapsto h_x(ix, 0)$$

is the Fourier transform of ν .

Proof. i) If the Laplace transform of ν is entire, then it is finite for all reals and is therefore in $\mathcal{B}^+(\mathbb{R}^+)$. Conversely, if $\nu \in \mathcal{B}^+(\mathbb{R}^+)$ then its Laplace transform is finite everywhere and ν has moments of all orders. The rest of part (i) follows (see, for example, Dybvig and Rogers (1997, Lemma 1 in the Appendix)).

ii) Using (27), we differentiate under the integral sign (justified using the dominated convergence theorem) to obtain

$$h_x(x, t) = \int_{-\infty}^{\infty} e^{yx - \frac{1}{2}y^2t} \nu(dy).$$

Thus $x \mapsto h_x(x, 0)$ is the Laplace transform of the measure ν . As $\nu \in \mathcal{B}^+(\mathbb{R}^+)$, we have by the first part of the Proposition that the Laplace transform is entire. In particular, its extension along the imaginary axis, $x \mapsto h_x(ix, 0)$, is the Fourier transform of ν . \square

We now recall in detail the one-to-one correspondence between positive and spatially strictly increasing solutions to (23) and spatially strictly increasing and strictly concave solutions to (21). The following result can be found in Musiela and Zariphopoulou (2010).

Proposition 3.6. i) Let h be a positive and spatially strictly increasing solution to (23) and let ν be the associated Borel measure (cf. (27)). If ν also satisfies $\nu((0, 1]) = 0$ and $\int_{1+}^{\infty} \frac{\nu(dy)}{y-1} < \infty$, then $u: (0, \infty) \times [0, \infty) \rightarrow \mathbb{R}$ is given by

$$u(x, t) = -\frac{1}{2} \int_0^t e^{-h^{(-1)}(x, s) + \frac{s}{2}} h_x \left(h^{(-1)}(x, s), s \right) ds + \int_0^x e^{-h^{(-1)}(z, 0)} dz \quad (28)$$

and satisfies

$$\lim_{x \rightarrow 0} u(x, t) = 0, \quad \text{for } t \geq 0. \quad (29)$$

On the other hand, if $\nu((0, 1]) > 0$ and/or $\int_{1+}^{\infty} \frac{\nu(dy)}{y-1} = \infty$, then

$$u(x, t) = -\frac{1}{2} \int_0^t e^{-h^{(-1)}(x, s) + \frac{s}{2}} h_x \left(h^{(-1)}(x, s), s \right) ds + \int_{\hat{x}}^x e^{-h^{(-1)}(z, 0)} dz, \quad (30)$$

for $\hat{x} > 0$ with

$$\lim_{x \rightarrow 0} u(x, t) = -\infty, \quad \text{for } t \geq 0. \quad (31)$$

For each $t \geq 0$, the Inada conditions

$$\lim_{x \rightarrow 0} u_x(x, t) = \infty \quad \text{and} \quad \lim_{x \rightarrow \infty} u_x(x, t) = 0 \quad (32)$$

are satisfied for both (28) and (30), respectively.

ii) Conversely, let $u: (0, \infty) \times [0, \infty) \rightarrow \mathbb{R}$ be spatially strictly increasing and strictly concave and satisfy (21) as well as the Inada conditions (32). If u satisfies (29), then there exists $\nu \in \mathcal{B}^+(\mathbb{R}^+)$ satisfying $\nu((0, 1]) = 0$ and $\int_{1+}^{\infty} \frac{\nu(dy)}{y-1} < \infty$ such that u admits representation (28) with h given by (27). On the other hand, if u satisfies (31), then there exists $\nu \in \mathcal{B}^+(\mathbb{R}^+)$ and either (i) $\nu((0, 1]) > 0$, or (ii) $\nu((0, 1]) = 0$ and $\int_{1+}^{\infty} \frac{\nu(dy)}{y-1} = \infty$, such that u admits representation (30) with h given by (27).

Remark 3.7. It follows from Proposition 3.6 that there exists a monotone forward investment process with initial datum u_0 if and only if the initial condition $h(x, 0)$ for the space-time harmonic function h , associated to u via (22), is given by

$$h(x, 0) = \int_{0+}^{\infty} \frac{e^{yx}}{y} \nu(dy),$$

for some $\nu \in \mathcal{B}^+(\mathbb{R}^+)$. Therefore, the set of initial conditions for h and, thus of u , is restricted to be those functions representable as a particular integral with respect to a Borel measure with finite Laplace transform.

3.2.3 Solution to the model under monotone forward investment performance criteria

We are now ready to recall the characterization of all objects of interest in the case of the monotone forward investment performance criterion. Note that we introduce condition (33), which is a stronger condition than is needed for the representations of h (cf. (26)) and thus of u , but is sufficient to guarantee the admissibility of the candidate optimal policy (35). The following result can be found in Musiela and Zariphopoulou (2010).

Theorem 3.1. *i) Let h be a positive and spatially strictly increasing solution to (23), for $(x, t) \in \mathbb{R} \times [0, \infty)$, and assume that the associated measure ν satisfies*

$$\int_{-\infty}^{\infty} e^{yx + \frac{1}{2}y^2t} \nu(dy) < \infty, \quad (x, t) \in \mathbb{R} \times [0, \infty). \quad (33)$$

Let A_t and M_t , $t \geq 0$, be as in (12) and define the processes X_t^ and π_t^* by*

$$X_t^* = h \left(h^{(-1)}(x_0, 0) + A_t + M_t, A_t \right) \quad (34)$$

and

$$\pi_t^* = h_x \left(h^{(-1)}(X_t^*, A_t), A_t \right) \sigma^{(-1)}(t) \lambda(t), \quad (35)$$

for $t \geq 0$, $x_0 > 0$, with h as above and $h^{(-1)}$ being its spatial inverse. Then, the portfolio process π_t^ is admissible and generates X_t^* , i.e.*

$$X_t^* = x_0 + \int_0^t \sigma(s) \pi_s^* \cdot (\lambda(s) ds + dW_s).$$

ii) Let u be a spatially strictly increasing and strictly concave solution to (21), associated to h via Proposition 3.6. Let $U(x, t)$, $t \geq 0$, $x > 0$ be given by

$$U(x, t) = u(x, A_t). \quad (36)$$

Then $U(x, t)$ is a forward investment performance process and the processes X_t^ and π_t^* defined in (34) and (35) are optimal.*

Remark 3.8. The measure ν encapsulates the investor's risk preferences under monotone forward investment performance criteria. To see this, recall that in the expected utility framework, the investor's initial wealth, market input, and her terminal utility function comprise the set of inputs that are sufficient to solve the investment problem (see Proposition 3.1). On the other hand, under monotone forward investment criteria the sufficient set of inputs is composed of the investor's initial wealth, market input and an admissible Borel measure ν (rather than a utility function). Indeed, given an admissible measure ν , one forms the function h via (27) and the function u via Proposition 3.6 (ν also determines the initial datum u_0 ; see Remark 3.7). In turn, one forms the investor's optimal policy and forward investment performance process using Theorem 3.1.

To close this section, we present the following scaling result, which shows that one can normalize the function h and assume that the measure ν is a finite Borel measure of arbitrary total mass. This fact will be used in the proof of Theorem 5.1. To this end, we denote by h_0 the total mass of ν and, with a slight abuse of notation, the associated wealth process by $X_t^*(x_0; h_0), t \geq 0$.

Proposition 3.9. *For $h_0 = \nu(\mathbb{R})$, the optimal wealth process satisfies, for $t \geq 0$,*

$$\frac{k_0}{h_0} X_t^*(x_0; h_0) = X_t^* \left(\frac{k_0}{h_0} x_0; k_0 \right),$$

where k_0 is an arbitrary positive constant.

Proof. Let $\hat{h}(x, t) = \frac{k_0}{h_0} h(x, t)$. Then,

$$\begin{aligned} X_t^*(x_0; h_0) &= h \left(h^{(-1)}(x_0, 0) + A_t + M_t, A_t \right) = \frac{h_0}{k_0} \hat{h} \left(h^{(-1)}(x_0, 0) + A_t + M_t, A_t \right) \\ &= \frac{h_0}{k_0} \hat{h} \left(\hat{h}^{(-1)} \left(\frac{k_0}{h_0} x_0, 0 \right) + A_t + M_t, A_t \right) = \frac{h_0}{k_0} X_t^* \left(\frac{k_0}{h_0} x_0; k_0 \right), \end{aligned}$$

where we have used the fact that $h^{(-1)}(x_0, 0) = \hat{h}^{(-1)} \left(\frac{k_0}{h_0} x_0, 0 \right)$. \square

4 Targeted wealth distributions in a fixed investment horizon setting

In this section we consider a Merton investor with the *fixed* investment horizon $[0, T]$, for some arbitrary positive terminal time $T < \infty$. The investment horizon is preset at initial time, when investment begins, and does not change throughout the course of investing. First, we present the case where the investor chooses a probability distribution for her terminal wealth. Subsequently, we consider an investor who chooses a probability distribution for her wealth to be realized at some arbitrary intermediate time within her investment horizon. In both cases, we show how, for a given initial wealth $x_0 > 0$, the investor's targeted distribution and an estimate of the market price of risk can be used to:

- determine if the chosen distribution is attainable in this market environment;

- infer the investor's risk preferences; and
- describe how the investor should invest to attain her goal.

We start with the family of distributions that we consider herein. Throughout, the function $\Phi: \mathbb{R} \rightarrow (0, 1)$ denotes the distribution function of the standard normal random variable.

Assumption 3. A chosen distribution function $F: (0, \infty) \rightarrow (0, 1)$ for future wealth is continuous, strictly increasing, and satisfies

$$F^{(-1)}(\Phi(x)) \leq K e^{a|x|}, \quad x \in \mathbb{R}, \quad (37)$$

for some positive constants K and a .

4.1 Investment target placed at terminal time

We start with the case in which the investor specifies a desired distribution for her terminal wealth. We address the three bullet points above. With regards to the second point, we infer the investor's risk preferences by finding her marginal utility function.

Theorem 4.1. *Suppose the investor with initial wealth $x_0 > 0$ targets her terminal wealth X_T^* to have distribution function F satisfying Assumption 3. Let A_t and M_t , $t \in [0, T]$, be as in (12). Then, the following hold.*

i) *The investor's target can be attained only if F satisfies the budget constraint*

$$x_0 = \frac{1}{\sqrt{2\pi A_T}} \int_{-\infty}^{\infty} e^{-\frac{y^2}{2A_T}} F^{(-1)}\left(\Phi\left(\frac{y - A_T}{\sqrt{A_T}}\right)\right) dy, \quad (38)$$

where $F^{(-1)}$ denotes the inverse of F .

ii) *If F satisfies (38), then the investor's marginal utility function is given by*

$$U_T'(x) = \exp\left(-\sqrt{A_T} \Phi^{(-1)}(F(x))\right). \quad (39)$$

iii) *The investor's optimal wealth and portfolio processes are given, respectively, by*

$$X_t^* = h\left(h^{(-1)}(x_0, 0) + A_t + M_t, t\right), \quad t \in [0, T], \quad (40)$$

and

$$\pi_t^* = h_x\left(h^{(-1)}(X_t^*, t), t\right) \sigma^{(-1)}(t) \lambda(t), \quad t \in [0, T], \quad (41)$$

where the function h is given by

$$h(x, t) = \frac{1}{\sqrt{2\pi(A_T - A_t)}} \int_{-\infty}^{\infty} e^{-\frac{1}{2} \frac{(x-y)^2}{(A_T - A_t)}} F^{(-1)} \left(\Phi \left(\frac{y}{\sqrt{A_T}} \right) \right) dy. \quad (42)$$

Proof. If F is the desired wealth distribution function, then (18) yields

$$\begin{aligned} F(y) &= \mathbb{P}(X_T^* \leq y) = \mathbb{P} \left(h(h^{(-1)}(x_0, 0) + M_T + A_T, T) \leq y \right) \\ &= \mathbb{P} \left(M_T \leq h^{(-1)}(y, T) - h^{(-1)}(x_0, 0) - A_T \right) \\ &= \Phi \left(\frac{h^{(-1)}(y, T) - h^{(-1)}(x_0, 0) - A_T}{\sqrt{A_T}} \right), \end{aligned} \quad (43)$$

where we used that M_T is centered normal with variance A_T (see (12)).

Next, we choose

$$h^{(-1)}(x_0, 0) = -A_T, \quad (44)$$

which, as we explain in detail in Remark 4.1, can be done without loss of generality.

From the above and (43), we then find that

$$h(x, T) = F^{(-1)} \left(\Phi \left(\frac{x}{\sqrt{A_T}} \right) \right), \quad x \in \mathbb{R}. \quad (45)$$

To show i), observe that from (18), (44), and (45) we have

$$X_T^* = F^{(-1)} \left(\Phi \left(\frac{M_T}{\sqrt{A_T}} \right) \right).$$

On the other hand, it is well known (see, for example, Karatzas and Shreve (1998)) that the budget constraint $x_0 = \mathbb{E}(Z_T X_T^*)$, where Z_T is as in (13), is binding. Combining the above, we deduce that

$$x_0 = \frac{1}{\sqrt{2\pi A_T}} \int_{-\infty}^{\infty} e^{-\frac{y^2}{2A_T}} F^{(-1)} \left(\Phi \left(\frac{y - A_T}{\sqrt{A_T}} \right) \right) dy.$$

Recall that F satisfies the inequality (37) and, therefore, the above integral converges.

To prove ii), we use equality (45) and the terminal condition for h from (17) to obtain

$$I_T(e^{-x}) = F^{(-1)} \left(\Phi \left(\frac{x}{\sqrt{A_T}} \right) \right), \quad x \in \mathbb{R}. \quad (46)$$

Since $I_T = (U_T')^{(-1)}$, we have that

$$U_T'(x) = \exp \left(-\sqrt{A_T} \Phi^{(-1)}(F(x)) \right), \quad (47)$$

and (39) follows.

We note that the conditions $\lim_{x \downarrow 0} F(x) = 0$ and $\lim_{x \uparrow \infty} F(x) = 1$ on F ensure that U_T satisfies the Inada conditions (15). Moreover, the polynomial growth requirement (16) on I necessitates the condition

$$F^{(-1)}(\Phi(x)) \leq a + ae^{b|x|}, \quad x \in \mathbb{R}, \quad (48)$$

for some positive constants a and b , for which (37) is sufficient.

Finally, to show iii), we recall that the function h satisfies (17). Replacing the terminal condition with (46) and using the representation formula for the solution of the Cauchy problem, we obtain (42). □

Remark 4.1. It is well known that an expected utility maximizer's optimal wealth process is invariant under positively-sloped linear transformations of the utility function U_T . This fact leads to a crucial observation used in the proof of Theorem 4.1, namely that the constant $h^{(-1)}(x_0, 0)$ can be chosen, without loss of generality, to be any real number. To see this, suppose that the investor has utility function U_T . Let $I_T = (U'_T)^{(-1)}$ and solve (17) to obtain h , and suppose that $h^{(-1)}(x_0, 0) = c_1 \in \mathbb{R}$. Now let $\tilde{U}_T(x) = e^{c_1 - c_2} U_T(x)$, for some $c_2 \in \mathbb{R}, c_2 \neq c_1$, be a positively-sloped linear transformation of U_T . Next, let $\tilde{I}_T(y) = (\tilde{U}'_T)^{(-1)}$ and let \tilde{h} be the solution to (17) using \tilde{I}_T in the terminal condition. It is then easily seen that $\tilde{I}_T(y) = I_T(e^{c_2 - c_1} y)$ and, in turn, that $\tilde{h}(x, t) = h(x + c_1 - c_2, t)$. From this, one observes that the investor's optimal wealth process is invariant under this transformation, that is, using (18), we have

$$X_t^* = h\left(h^{(-1)}(x_0, 0) + A_t + M_t, t\right) = \tilde{h}\left(\tilde{h}^{(-1)}(x_0, 0) + A_t + M_t, t\right), \quad t \in [0, T],$$

where A_t and $M_t, t \in [0, T]$, are as in (12). Moreover, one obtains that $\tilde{h}^{(-1)}(x_0, 0) = c_2$, and we easily conclude.

Remark 4.2. Recall that in the works of Sharpe *et al.* (see Sharpe *et al.* (2000), Sharpe (2001), and Goldstein *et al.* (2008)) the market is incomplete. As mentioned in section 2, the developers of *The Distribution Builder* introduce the additional assumption that

the investor wants to achieve her distribution in a cost-efficient manner, in that any other investment strategy that achieves the desired distribution costs at least as much. This cost-efficiency property is guaranteed, however, in our complete market setting with an expected utility maximizer over terminal wealth (see Bernard *et al.* (2012), Dybvig (1988a) and Dybvig (1988b)). Indeed, a straightforward change of variables shows that the budget constraint (38) can be rewritten as

$$x_0 = \int_0^1 F_{Z_T}^{(-1)}(y) F^{(-1)}(1-y) dy, \quad (49)$$

where F_{Z_T} is the distribution function of the state price density Z_T defined in (13) and F is the investor's desired distribution function as in Theorem 4.1. The significance of this is that the right-hand side of (49) is known to be the *distributional price* (see Dybvig (1988a)), of the distribution F in the given market. That is, among all \mathcal{F}_T -measurable random variables X_T^π with distribution function F that can be achieved using a strategy $\pi \in \mathcal{A}_T$, the one requiring the least initial endowment is given by the right-hand side of (49). Thus, the investor who maximizes her expected utility also achieves her distributional price.

Example 1. Suppose the investor aims at acquiring lognormally distributed terminal wealth, i.e. $\log X_T^*$ is centered normal with variance b for some parameter $b > 0$. Note that, initially, this choice does not specify a single distribution, but rather a family of distributions parameterized by b . The budget constraint (38) then determines the unique b that is consistent with the investor's choice and utility criterion. To this end, it is easily seen that the inequality (37) is satisfied, and therefore (38) yields that

$$x_0 = \frac{1}{\sqrt{2\pi A_T}} \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2A_T} y^2 + \sqrt{\frac{b}{A_T}} y - \sqrt{b A_T}\right) dy = \exp\left(\frac{b}{2} - \sqrt{b A_T}\right). \quad (50)$$

Straightforward manipulation of (50) yields the following necessary relationship between the investor's wealth and the market, namely

$$A_T + 2 \log x_0 = b - 2\sqrt{b A_T} + A_T = \left(\sqrt{b} - \sqrt{A_T}\right)^2 \geq 0,$$

which, in turn, yields that $b = \left(\sqrt{A_T} + \sqrt{A_T + 2 \log x_0}\right)^2$.

From (39), we deduce the investor's marginal utility function,

$$U_T'(x) = x^{-\frac{1}{\beta}}, \quad \text{with} \quad \beta := 1 + \left|1 - \sqrt{\frac{b}{A_T}}\right|.$$

Therefore, we have two cases for the investor's utility function:

(a) If $\beta > 1$, then

$$U_T(x) = \frac{1}{1 - \frac{1}{\beta}} x^{1 - \frac{1}{\beta}}.$$

(b) If $\beta = 1$, then $U_T(x) = \log x$.

The underlying harmonic function (see (42)) is then given by

$$h(x, t) = \exp \left(\beta x + \frac{1}{2} \beta^2 (A_T - A_t) \right)$$

and, in turn, (40) and (41) yield the optimal policies

$$X_t^* = x_0 e^{(\beta - \frac{1}{2} \beta^2) A_t + \beta M_t} \quad \text{and} \quad \pi_t^* = \beta x_0 e^{(\beta - \frac{1}{2} \beta^2) A_t + \beta M_t} \sigma^{(-1)}(t) \lambda(t).$$

Example 2. Suppose the investor with initial wealth $x_0 > 3$ targets that, if X_T^* is her terminal wealth, then the random variable $g(X_T^*)$ has is centered normal with variance b for some $b > A_T$, where

$$g(x) = \log(-1 + \sqrt{1 + x}), \quad x \in (0, \infty).$$

As in the previous example, this specifies only a family of distributions, and the parameter b is determined through the budget constraint as follows. We have that $F^{(-1)}(\Phi(x)) = \exp(2\sqrt{bx}) + 2\exp(\sqrt{bx})$, and so the inequality (37) is satisfied. The budget constraint (38) then shows the implicit relationship between the parameter b in terms of x_0 and A_T , namely

$$x_0 = \exp \left(2(b - \sqrt{bA_T}) \right) + 2 \exp \left(\frac{b}{2} - \sqrt{bA_T} \right). \quad (51)$$

It is easily seen that there is a unique b that satisfies (51) under our assumptions. From (39), the investor's marginal utility function is given by

$$U_T'(x) = (-1 + \sqrt{1 + x})^{-\sqrt{\frac{A_T}{b}}}.$$

The underlying harmonic function in (42) is

$$h(x, t) = \left(\exp \left(2\sqrt{\frac{b}{A_T}} x + 2\frac{b}{A_T} (A_T - A_t) \right) + 2 \exp \left(\sqrt{\frac{b}{A_T}} x + \frac{1}{2} \frac{b}{A_T} (A_T - A_t) \right) \right).$$

Using the above and (40) and (41), one can find the optimal wealth and portfolio processes.

4.2 Investment target placed at an intermediate investment time

In Theorem 4.1, we showed that a Merton investor who specifies her desired distribution for wealth at terminal time T will effectively determine her risk preferences at terminal time, and, in turn, the optimal policy throughout. Next, we consider an investor who specifies a distribution for her wealth to be realized at some arbitrary, but fixed, intermediate time $\hat{T} \in (0, T)$.

As in Theorem 4.1, we find that the specification of this single distribution at time \hat{T} , when combined with the investor's initial wealth and market input, is sufficient to determine the feasibility of the desired distribution, the optimal policies that achieve the investor's goal, and the investor's risk preferences. The proof relies on the results of Widder on the inversion of the Weierstrass transform (see Hirschman and Widder (1955)).

Before we proceed, we introduce some additional technical assumptions on the investor's chosen distribution.

Assumption 4. Let $F: (0, \infty) \rightarrow (0, 1)$ be a chosen wealth distribution function. Let $\hat{T} \in (0, T)$ and recall the function $A_t, t \in [0, T]$, in (12). Define the function $G: \mathbb{R} \rightarrow (0, \infty)$ associated to F by

$$G(x) := F^{(-1)} \left(\Phi \left(\frac{cx}{\sqrt{A_{\hat{T}}}} \right) \right), \quad \text{with } c := \sqrt{\frac{A_T - A_{\hat{T}}}{2}}. \quad (52)$$

We assume that:

- (i) G extends analytically to an entire function on \mathbb{C} ;
- (ii) G satisfies the growth condition

$$\limsup_{|y| \rightarrow \infty} \frac{|G(x + iy)|}{|y|e^{y^2/4}} = 0, \quad \text{uniformly on closed subintervals of } \mathbb{R} \text{ containing } x;$$

- (iii) The function $g: \mathbb{R} \times (0, 1) \rightarrow \mathbb{C}$ defined by

$$g(x, t) := \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-y^2/4t} G(x + iy) dy \quad (53)$$

is real-valued and nonnegative for all $(x, t) \in \mathbb{R} \times (0, 1)$.

We are now ready to state the results. We recall that $I_T: (0, \infty) \rightarrow (0, \infty)$ is the inverse of the investor's marginal utility function $U'_T: (0, \infty) \rightarrow (0, \infty)$, and that $\Phi: \mathbb{R} \rightarrow (0, 1)$ denotes the distribution function of the standard normal random variable.

Theorem 4.2. *Suppose the investor with initial wealth $x_0 > 0$ targets her wealth $X_{\hat{T}}^*$, at some intermediate time $\hat{T} \in (0, T)$, to have distribution function F satisfying Assumption 3. Let A_t and M_t , $t \in [0, T]$, be as in (12). Then, the following hold.*

i) *The investor's target can be attained only if F satisfies the budget constraint*

$$x_0 = \frac{1}{\sqrt{2\pi A_{\hat{T}}}} \int_{-\infty}^{\infty} e^{-\frac{y^2}{2A_{\hat{T}}}} F^{(-1)} \left(\Phi \left(\frac{y - A_{\hat{T}}}{\sqrt{A_{\hat{T}}}} \right) \right) dy. \quad (54)$$

ii) *If F satisfies (54) and, in addition, Assumption 4, then the inverse I_T of the investor's marginal utility function is given by*

$$I_T(x) = \frac{1}{\sqrt{2\pi(A_T - A_{\hat{T}})}} \int_{-\infty}^{\infty} e^{-\frac{1}{2} \frac{y^2}{(A_T - A_{\hat{T}})}} F^{(-1)} \left(\Phi \left(\frac{-\log x + iy}{\sqrt{A_{\hat{T}}}} \right) \right) dy. \quad (55)$$

iii) *If F satisfies (54) and, in addition, Assumption 4, then the investor's optimal wealth and portfolio processes are given by*

$$X_t^* = h \left(h^{(-1)}(x_0, 0) + A_t + M_t, t \right), \quad t \in [0, T], \quad (56)$$

and

$$\pi_t^* = h_x \left(h^{(-1)}(X_t^*, t), t \right) \sigma^{(-1)}(t) \lambda(t), \quad t \in [0, T], \quad (57)$$

respectively, where the function h is given by

$$h(x, t) = \frac{1}{\sqrt{2\pi(A_T - A_t)}} \int_{-\infty}^{\infty} e^{-\frac{1}{2} \frac{(x-y)^2}{(A_T - A_t)}} I_T(e^{-y}) dy, \quad (58)$$

with I_T as in (55).

Proof. Recall that although the investor is specifying desired distributional data at time $\hat{T} \in (0, T)$, her investment horizon is $[0, T]$. If the investor targets her wealth at time \hat{T} to have distribution function F , then (18) yields

$$\begin{aligned} F(y) &= \mathbb{P}(X_{\hat{T}}^* \leq y) = \mathbb{P} \left(h \left(h^{(-1)}(x_0, 0) + M_{\hat{T}} + A_{\hat{T}}, \hat{T} \right) \leq y \right) \\ &= \Phi \left(\frac{h^{(-1)}(y, \hat{T}) - h^{(-1)}(x_0, 0) - A_{\hat{T}}}{\sqrt{A_{\hat{T}}}} \right), \end{aligned}$$

where we used that $M_{\hat{T}}$ is centered normal with variance $A_{\hat{T}}$. Inverting, we deduce that

$$h(x, \hat{T}) = F^{(-1)} \left(\Phi \left(\frac{x}{\sqrt{A_{\hat{T}}}} \right) \right), \quad x \in \mathbb{R}, \quad (59)$$

where, in analogy to the proof of Theorem 4.1 and Remark 4.1, we have chosen

$$h^{(-1)}(x_0, 0) = -A_{\hat{T}}. \quad (60)$$

To show i), observe that from (18), (59), and (60) we have

$$X_{\hat{T}}^* = F^{(-1)} \left(\Phi \left(\frac{M_{\hat{T}}}{\sqrt{A_{\hat{T}}}} \right) \right). \quad (61)$$

Recall $Z_{\hat{T}}$ from (13). Then, (61) yields

$$x_0 = \mathbb{E}(Z_{\hat{T}} X_{\hat{T}}^*) = \frac{1}{\sqrt{2\pi A_{\hat{T}}}} \int_{-\infty}^{\infty} e^{-\frac{y^2}{2A_{\hat{T}}}} F^{(-1)} \left(\Phi \left(\frac{y - A_{\hat{T}}}{\sqrt{A_{\hat{T}}}} \right) \right) dy,$$

where the first equality is due to the well-known budget constraint in this model (see, for example, Karatzas and Shreve (1998)) and the fact that $Z_t X_t^*, t \in [0, T]$, is a \mathbb{P} -martingale. Recall that F satisfies the growth condition (37), and thus the above integral converges.

To prove ii), first note that by (59) and the uniqueness of the solution to (17), we must have

$$F^{(-1)} \left(\Phi \left(\frac{x}{\sqrt{A_{\hat{T}}}} \right) \right) = \frac{1}{\sqrt{2\pi(A_T - A_{\hat{T}})}} \int_{-\infty}^{\infty} e^{-\frac{1}{2} \frac{(x-y)^2}{(A_T - A_{\hat{T}})}} I_T(e^{-y}) dy, \quad x \in \mathbb{R}. \quad (62)$$

By a change of variables, we deduce that this is equivalent to

$$F^{(-1)} \left(\Phi \left(\frac{cx}{\sqrt{A_{\hat{T}}}} \right) \right) = \frac{1}{\sqrt{4\pi}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4} I_T(e^{-cy}) dy, \quad (63)$$

where $c := (\frac{1}{2}(A_T - A_{\hat{T}}))^{\frac{1}{2}}$. Next, we note that the right-hand side of (63) is the Weierstrass transform of the function $x \mapsto I(e^{-cx})$. By Hirschman and Widder (1955, Theorem 12.4 (p. 204); see also Definition 3.2 and Theorem 3.2 (p. 180)), for such a representation to exist and to converge for all $x \in \mathbb{R}$, it is necessary and sufficient that the function $G: \mathbb{R} \rightarrow (0, \infty)$, defined in (52), satisfies conditions (i), (ii) and (iii) of Assumption 4. Under these conditions, Widder's theorem on the inversion of the

Weierstrass transform (see Hirschman and Widder (1955, Theorem 7.4 p. 191)) yields that

$$I_T(e^{-cx}) = \lim_{t \uparrow 1} g(x, t) = \lim_{t \uparrow 1} \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-y^2/4t} G(x + iy) dy, \quad \text{a.e. } x \in \mathbb{R}, \quad (64)$$

with g as in (53). On the other hand, because both sides of (64) are continuous in x , this equality holds for all $x \in \mathbb{R}$. Moreover, since G satisfies the growth condition (ii) of Assumption 4, the integral in (64) is dominated by

$$\int_{-\infty}^{\infty} |y| e^{-y^2/4t} e^{-y^2/4} dy = \int_{-\infty}^{\infty} |y| e^{-\frac{(1+t)}{4t} y^2} dy \leq \int_{-\infty}^{\infty} |y| e^{-y^2/2} dy.$$

Since the dominant integral converges and is independent of t , we have by the dominated convergence theorem that

$$I_T(e^{-cx}) = \lim_{t \uparrow 1} g(x, t) = \frac{1}{\sqrt{4\pi}} \int_{-\infty}^{\infty} e^{-y^2/4} G(x + iy) dy,$$

which yields (55) after a change of variables.

Finally, part iii) follows from the representation formula for the solution of the Cauchy problem (17). \square

Example 3. Suppose the investor desires lognormally distributed wealth at time $\hat{T} \in (0, T)$, i.e. $\log X_{\hat{T}}^*$ is centered normal with variance b for some $b > 0$. As in Example 1, we note that this specifies only a family of distributions. The budget constraint (54) implies that

$$A_{\hat{T}} + 2 \log x_0 = b - 2\sqrt{bA_{\hat{T}}} + A_{\hat{T}} = \left(\sqrt{b} - \sqrt{A_{\hat{T}}}\right)^2 \geq 0,$$

and therefore, the distribution that is consistent with the investor's choice and criterion has parameter b given uniquely by

$$b = \left(\sqrt{A_{\hat{T}}} + \sqrt{A_{\hat{T}} + 2 \log x_0}\right)^2.$$

The function G (see (52)) then becomes

$$G(x) = e^{kx}, \quad \text{with} \quad k := \sqrt{\frac{A_T - A_{\hat{T}}}{2}} \left(1 + \left|1 - \sqrt{\frac{b}{A_{\hat{T}}}}\right|\right).$$

This function satisfies (i), (ii) and (iii) of Assumption 4.

Using (55), we easily see that the inverse of the investor's marginal utility is given by

$$I_T(e^{-x}) = e^{\beta x - k^2}, \quad \text{with} \quad \beta := 1 + \left| 1 - \sqrt{\frac{b}{A_{\hat{T}}}} \right|.$$

Therefore, the investor's marginal utility function is given by

$$U'_T(x) = e^{-\frac{1}{\beta}x - \frac{1}{\beta}},$$

while the underlying harmonic function (see (58)) is $h(x, t) = e^{-k^2} \exp(\beta x + \frac{1}{2}\beta^2(A_T - A_t))$.

Hence (56) and (57) yield the optimal policies

$$X_t^* = x_0 e^{(\beta - \frac{1}{2}\beta^2)A_t + \beta M_t} \quad \text{and} \quad \pi_t^* = \beta x_0 e^{(\beta - \frac{1}{2}\beta^2)A_t + \beta M_t} \sigma^{(-1)}(t) \lambda(t).$$

5 Targeted wealth distributions in a flexible horizon setting

We continue our study of how an investor's desired distribution for future wealth can be used to recover her risk preferences and construct her optimal policies. In the previous section, we considered a Merton investor with a fixed investment horizon $[0, T]$. In this section, we allow for the investor to have flexibility in her investment horizon. There are practical reasons for allowing such flexibility. For instance, the investor may not know *a priori* until when she will be investing, or may wish to invest indefinitely, or may wish have the flexibility to roll over her portfolio or otherwise extend her investment horizon beyond the original prespecified terminal time. Flexibility in the investment horizon falls outside the classical fixed-horizon Merton problem. An appropriate investment criterion is instead the forward investment performance framework, which we reviewed in section 3.2. Similar to the fixed horizon setting of section 4, we show how the investor's targeted distribution, her initial wealth, and an estimate of the market price of risk can be used to:

- determine if the chosen distribution is attainable in this market environment;
- infer the investor's risk preferences at initial time and describe how they change dynamically throughout the investment horizon; and

- describe how the investor should invest in the market to attain her goal.

5.1 Investment target at an arbitrary point for an investor without a fixed terminal horizon

We consider an investor in a flexible investment horizon setting who places a desired distribution for wealth at some fixed, but arbitrary, future time. The following result shows that the investor's desired distribution for future wealth, when combined with her initial wealth and market input, determines the Fourier transform of a Borel measure $\nu \in \mathcal{B}^+(\mathbb{R}^+)$, where $\mathcal{B}^+(\mathbb{R}^+)$ is as in (26). As discussed in section 3.2.1, this measure is the defining element for the functions u and h in the monotone forward investment performance framework. If, in addition, the measure satisfies (33), then one can also find the optimal wealth process, the optimal investment strategy π^* that achieves it, and the forward investment performance process via (34), (35), and (36), respectively.

We recall that the function $\Phi: \mathbb{R} \rightarrow (0, 1)$ denotes the distribution function of the standard normal random variable, and we denote by ϕ its density.

Theorem 5.1. *Suppose the investor with initial wealth $x_0 > 0$ targets her wealth X_T^* at some time $T \in (0, \infty)$ to have distribution function F satisfying Assumption 3. Let A_t and M_t , $t \geq 0$, be as in (12). Then, the following hold.*

i) *The investor's target can be attained only if F satisfies the budget constraint*

$$x_0 = \frac{1}{\sqrt{2\pi A_T}} \int_{-\infty}^{\infty} e^{-\frac{y^2}{2A_T}} F^{(-1)}\left(\Phi\left(\frac{y - A_T}{\sqrt{A_T}}\right)\right) dy. \quad (65)$$

ii) *If F satisfies (65), then the Fourier transform of the underlying measure ν is given by*

$$\varphi_\nu(x) = \frac{1}{A_T \sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{(ix-y)^2}{2A_T}} \frac{\phi\left(\frac{y}{\sqrt{A_T}}\right)}{f\left(F\left(\Phi\left(\frac{y}{\sqrt{A_T}}\right)\right)\right)} dy, \quad (66)$$

where f is the density of F . Moreover, if u_0 is the investor's initial datum, then

$$u'_0(x) = \exp\left(-h_0^{(-1)}(x)\right), \quad (67)$$

with h_0 given by

$$h_0(x) = \int_{0+}^{\infty} \frac{e^{yx}}{y} \nu(dy). \quad (68)$$

iii) If the above measure ν satisfies (33), then the investor's optimal wealth and portfolio processes are given by

$$X_t^* = h \left(h^{(-1)}(x_0, 0) + A_t + M_t, A_t \right), \quad t \geq 0, \quad (69)$$

and

$$\pi_t^* = h_x \left(h^{(-1)}(X_t^*, A_t), A_t \right) \sigma^{(-1)}(t) \lambda(t), \quad t \geq 0, \quad (70)$$

respectively, where h is given by

$$h(x, t) = \int_{0+}^{\infty} \frac{e^{yx - \frac{1}{2}y^2t}}{y} \nu(dy). \quad (71)$$

Proof. Let $h(x, t)$ be given by (27) for some $\nu \in \mathcal{B}^+(\mathbb{R}^+)$. Recall from Proposition 3.9 that we can assume, without loss of generality, that ν has arbitrary total mass. Therefore, we assume that ν is such that it satisfies $\int_{0+}^{\infty} \frac{e^{-A_T y}}{y} \nu(dy) = x_0$ or, equivalently, that

$$h^{(-1)}(x_0, 0) = -A_T. \quad (72)$$

Then, using (34), we obtain that

$$X_T^* = h(M_T, A_T). \quad (73)$$

If the investor targets her wealth at time T to have distribution function F , then using that M_T is centered normal with variance A_T , we deduce that

$$F(y) = \mathbb{P}(X_T^* \leq y) = \Phi \left(\frac{h^{(-1)}(y, A_T)}{\sqrt{A_T}} \right) \quad (74)$$

and, in turn, that

$$h(x, A_T) = F^{(-1)} \left(\Phi \left(\frac{x}{\sqrt{A_T}} \right) \right). \quad (75)$$

Part i) then follows from the well-known budget constraint in this model (see, for example, Karatzas and Shreve (1998)), (73), (75), (13), (72) and (37).

We now prove ii). By (23) and (75), the function h must solve

$$\begin{cases} h_t + \frac{1}{2}h_{xx} = 0, & (x, t) \in (0, \infty) \times (0, A_T) \\ h(x, A_T) = F^{(-1)} \left(\Phi \left(\frac{x}{\sqrt{A_T}} \right) \right), & x \in (0, \infty). \end{cases} \quad (76)$$

Condition (37) implies that the terminal data satisfies the Tychonov condition (see Friedman (1964, Chapter 1)) and so the unique solution to (76) is given by the convolution

$$h(x, t) = \frac{1}{\sqrt{2\pi(A_T - t)}} \int_{-\infty}^{\infty} e^{-\frac{1}{2} \frac{(x-y)^2}{(A_T - t)}} h(y, A_T) dy.$$

Since $x \mapsto h(x, A_T)$ is differentiable almost everywhere, we obtain

$$h_x(x, t) = \frac{1}{\sqrt{2\pi(A_T - t)}} \int_{-\infty}^{\infty} e^{-\frac{1}{2} \frac{(x-y)^2}{(A_T - t)}} h_x(y, A_T) dy.$$

By Proposition 3.5, we then conclude that the function $\varphi_\nu: \mathbb{R} \rightarrow \mathbb{C}$ given by

$$\begin{aligned} \varphi_\nu(x) &= h_x(ix, 0) \\ &= \frac{1}{\sqrt{2\pi A_T^2}} \int_{-\infty}^{\infty} e^{-\frac{(ix-y)^2}{2A_T}} \frac{\phi\left(\frac{y}{\sqrt{A_T}}\right)}{f\left(F\left(\Phi\left(\frac{y}{\sqrt{A_T}}\right)\right)\right)} dy \end{aligned}$$

is the Fourier transform of the implied measure ν . Equations (67) and (68) then follow from (21), (22), and Proposition 3.3.

Part iii) follows by Theorem 3.1 and (27). □

Remark 5.1. The growth assumption (37) for the distribution F in Assumption 3 can be slightly relaxed. Indeed, in order for the Tychonov condition to be satisfied in (76), it is sufficient that

$$F^{(-1)}(\Phi(x)) \leq K e^{ax^2}, \quad x \in \mathbb{R}, \quad (77)$$

for some positive constants K and $a < \frac{1}{2}$. In Example 6, we analyze a case in which F satisfies (77) but not (37).

Example 4. Suppose that the investor desires lognormally distributed wealth at time T , i.e. $\log X_T^*$ is centered normal with variance b for some $b > 0$. Working as in the previous examples, in order to specify the distribution that is consistent with the investor's choice and criterion, we use the budget constraint (65) to find that

$$A_T + 2 \log x_0 = b - 2\sqrt{bA_T} + A_T = \left(\sqrt{b} - \sqrt{A_T}\right)^2 \geq 0.$$

Thus, b is given uniquely by

$$b = \left(\sqrt{A_T} + \sqrt{A_T + 2 \log x_0}\right)^2.$$

Using this and (66), the Fourier transform of ν is then given by

$$\varphi_\nu(x) = \beta \exp(ix\beta), \quad \text{with} \quad \beta := 1 + \left| 1 - \sqrt{\frac{b}{A_T}} \right|.$$

We easily see that this is the Fourier transform of the Dirac point mass $\nu = \beta\delta_\beta$, which satisfies the admissibility condition (33). Using (67) and (68), we find that $u'_0(x) = x^{-\frac{1}{\beta}}$ and, using (27), we deduce that $h(x, t) = e^{\beta x - \frac{1}{2}\beta^2 t}$.

The associated optimal wealth and portfolio processes are given by

$$X_t^* = x_0 e^{(\beta - \frac{1}{2}\beta^2)A_t + \beta M_t} \quad \text{and} \quad \pi_t^* = \beta x_0 e^{(\beta - \frac{1}{2}\beta^2)A_t + \beta M_t} \sigma^{(-1)}(t) \lambda(t).$$

Finally, we deduce the investor's forward investment performance process. If, for instance, $\beta > 1$, the investor's forward investment performance is given by

$$U(x, t) = \frac{\beta^{\frac{\beta-1}{\beta}}}{\beta-1} x^{\frac{\beta-1}{\beta}} e^{-\frac{\beta-1}{2}A_t}.$$

Example 5. Suppose that the investor with initial wealth $x_0 > 3$ desires that, if X_T^* is her wealth at time T , then the random variable $g(X_T^*)$ is centered normal with variance b for some $b > A_T$, where $g: (0, \infty) \rightarrow \mathbb{R}$ is given by $g(x) = \log(-1 + \sqrt{1+x})$. Again, note that this is a family of distributions. Using the budget constraint (65) we find that

$$x_0 = \exp\left(2\left(b - \sqrt{A_T b}\right)\right) + 2 \exp\left(\frac{b}{2} - \sqrt{A_T b}\right). \quad (78)$$

Under our assumptions, it is easily seen that there is a unique b that satisfies (78).

Next, the Fourier transform of the implied measure ν is found via (66). Specifically,

$$\begin{aligned} \varphi_\nu(x) &= 2\sqrt{\frac{b}{A_T}} \left(\exp\left(2ix\sqrt{\frac{b}{A_T}} + 2b - 2\sqrt{bA_T}\right) + \exp\left(ix\sqrt{\frac{b}{A_T}} + \frac{b}{2} + \sqrt{bA_T}\right) \right) \\ &= \sqrt{\frac{b}{A_T}} \left(k_1 \exp\left(2ix\sqrt{\frac{b}{A_T}}\right) + k_2 \exp\left(ix\sqrt{\frac{b}{A_T}}\right) \right), \end{aligned}$$

where the constants k_1 and k_2 are given by

$$k_1 = \frac{2e^{2b-2\sqrt{bA_T}}}{e^{2b-2\sqrt{bA_T}} + 2e^{b/2-\sqrt{bA_T}}} \quad \text{and} \quad k_2 = \frac{2e^{b/2-\sqrt{bA_T}}}{e^{2b-2\sqrt{bA_T}} + 2e^{b/2-\sqrt{bA_T}}}.$$

The implied measure ν is then given by the sum of Dirac point masses:

$$\nu = k_1 \beta \delta_{2\beta} + k_2 \beta \delta_\beta, \quad \text{with} \quad \beta = \sqrt{\frac{b}{A_T}}.$$

Using (68) and (67), we in turn deduce that

$$u'_0(x) = \left(\sqrt{\frac{2}{k_1}x + \frac{k_2^2}{k_1^2}} - \frac{k_2}{k_1} \right)^{-\frac{1}{\beta}}.$$

Moreover, it easily follows that ν satisfies (33). Using (71) we then find

$$h(x, t) = \frac{k_1}{2} e^{2\beta x - 2\beta^2 t} + k_2 e^{\beta x - \frac{1}{2}\beta^2 t}.$$

From there, one can apply formulae (69), (70), and (36), to find the optimal wealth process, the optimal investment policy that generates it, and the forward investment performance process that are consistent with the investor's preferences.

We conclude this section by considering one case where the range of the investor's wealth is the entire real line. Although we do not systematically consider investment problems in which wealth can become negative herein, we nevertheless provide an informal example. It was shown in Musiela and Zariphopoulou (2010) that representation results for the optimal policies and the forward investment performance process in terms of a Borel measure ν hold in the case of possibly negative wealth. These representation results are similar to those in subsection 5.1 in the case of nonnegative wealth.

Example 6. Suppose the investor targets her wealth at time T to have distribution function

$$F(y) = \Phi \left(\sqrt{1 + 1/A_T} H^{(-1)}(y) \right),$$

where Φ is the standard normal distribution function, and $H: \mathbb{R} \rightarrow \mathbb{R}$ is given by

$$H(x) = \int_0^x e^{\frac{1}{2}z^2} dz.$$

We assume that the investor's initial wealth is such that the budget constraint (65) is satisfied. Note that F satisfies the growth condition (77) but not (37). Nevertheless, as mentioned in Remark 5.1, the conclusions of Theorem 5.1 hold.

After some tedious but straightforward calculations, we deduce via (66) that the Fourier transform of the implied measure ν is given by

$$\varphi_\nu(x) = e^{-\frac{1}{2}x^2}.$$

This is the characteristic function of a standard normal random variable, and so $\nu(dy) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} dy$. Note, however, that this measure ν violates condition (33) for $t > 0$ and satisfies only (26). In this situation, one can work with the so-called local forward investment performance process (see Musiela and Zariphopoulou (2010, section 2)).

6 Comments and conclusions

6.1 Time-consistency of distributional investment targets

Besides the feasibility conditions we considered in sections 4 and 5, it is natural to investigate whether an investor who desires a certain wealth profile at time T_1 can also choose a wealth profile at a different time $T_2, T_1 \neq T_2$. The market model considered herein, however, is not general enough to allow for this to be done in an arbitrary way. Indeed, Theorems 4.1, 4.2, and 5.1 demonstrate that, along with the investor's initial wealth and market input, the specification of a *single* desired distribution for future wealth fully determines the investor's optimal wealth process at all times within the investor's investment horizon. Hence, the investor in the market considered herein is only permitted to choose a distribution for wealth at one future time, in both the fixed and flexible horizon settings. This choice determines her wealth process pointwise, and thus in distribution, at all other times.

6.2 Role of initial wealth

The investor's initial wealth x_0 plays an important, albeit subtle, role in our work. The choice of x_0 is arbitrary but fixed throughout the paper. The initial wealth, together with the investor's choice of distribution and market input, comprises the set of necessary inputs for the analysis. Indeed, the three inputs are interrelated via the budget constraints (see (38), (54) and (65)). Therefore, the set of distributions attainable in a given market environment depend strongly on the investor's initial wealth; varying the initial wealth generally results in a different set of attainable distributions.

6.3 Conclusions and future directions

Sharpe *et al.* proposed the idea of having an expected utility maximizer choose a probability distribution for future wealth as an input to her investment problem instead of a utility function. The essence of their method is that an investor selects a desired probability distribution for future wealth and, subject to her initial wealth and market constraints, is then told the optimal policies and risk preferences consistent with that choice. We extended this normative approach to a continuous-time complete market framework with variable market coefficients. This results in added flexibility as to when the investor would like to realize her desired distribution as well as flexibility with the investment horizon itself.

Our method relies on being able to estimate the market price of risk, and one possible direction for future work is to address how to formulate and solve similar questions in a complete or incomplete market with stochastic market coefficients. We have also seen that the investor cannot arbitrarily choose multiple distributions for future wealth throughout the investment horizon in the model considered herein, regardless of whether she is a Merton investor or a forward investor with monotone performance criteria. Perhaps the selection of multiple distributions for future wealth can be done in a more general market model. Finally, another extension would be to consider a multi-period model, in the sense that the investor places a distribution for wealth at some future time T_1 , invests optimally on $[0, T_1]$, and then at time T_1 selects another distribution for wealth to be placed at time $T_2 > T_1$, having realized her wealth random variable at T_1 according to the previously chosen distribution. These are all subjects of ongoing research.

Acknowledgements

The author would like to thank T. Zariphopoulou, M. Sîrbu, and G. Žitković for their comments and suggestions.

Appendix: Proof of Proposition 3.1

For completeness, we provide the proof of Proposition 3.1, which is an adaptation of the result of Källblad (2011) for the case of constant coefficients.

Proof. Under Assumptions 1 and 2, it is well known (see, for example, Karatzas and Shreve (1998)) that the optimal wealth process is given by $X_t^* = \psi(kZ_t, t)$, where the function $\psi: (0, \infty) \times [0, T] \rightarrow (0, \infty)$ is defined by

$$\psi(y, t) = \mathbb{E}_{\mathbb{Q}} \left[I_T \left(y \frac{Z_T}{Z_t} \right) \right].$$

Herein, $\mathbb{E}_{\mathbb{Q}}$ denotes expectation under the equivalent martingale measure \mathbb{Q}^T given by $\frac{d\mathbb{Q}^T}{d\mathbb{P}} = Z_T$ where Z_T is as in (13), while the Lagrange multiplier $k > 0$ is the solution to

$$\mathbb{E}[Z_T I_T(kZ_T)] = x_0. \quad (79)$$

Moreover, by the polynomial growth assumption (16) on I_T and the Hölder continuity of $|\lambda(t)|$, it is known (see Karatzas and Shreve (1998, Lemma 8.4 (p. 122))) that $\psi \in C((0, \infty) \times [0, T]) \cap C^{2,1}((0, \infty) \times [0, T])$ and solves the Cauchy problem

$$\begin{cases} \psi_t(y, t) + \frac{1}{2} |\lambda(t)|^2 y^2 \psi_{yy}(y, t) + |\lambda(t)|^2 y \psi_y(y, t) = 0, & (y, t) \in (0, \infty) \times [0, T] \\ \psi(y, T) = I_T(y), & y \in (0, \infty), \end{cases}$$

and that, for each $t \in [0, T)$, the function $y \mapsto \psi(y, t)$ is strictly decreasing.

Next, we define a function $h: \mathbb{R} \times [0, T] \rightarrow (0, \infty)$ by

$$h(x, t) := \psi(e^{-x + \frac{1}{2} A_t}, t),$$

where A_t is as in (12). Then

$$h(x, T) = I_T \left(e^{-x + \frac{1}{2} A_T} \right). \quad (80)$$

Since the investor's optimal strategy is invariant under positive dilations of the argument of $I_T(\cdot)$ (by Remark 4.1), we can assume the terminal condition is $h(x, T) = I_T(e^{-x})$. We then have that $h \in C(\mathbb{R} \times [0, A_T]) \cap C^{2,1}(\mathbb{R} \times [0, A_T])$ and solves

$$\begin{cases} h_t(x, t) + \frac{1}{2} |\lambda(t)|^2 h_{xx}(x, t) = 0, & (x, t) \in \mathbb{R} \times [0, T] \\ h(x, T) = I_T(e^{-x}), & x \in \mathbb{R}. \end{cases} \quad (81)$$

Let $h^{(-1)}$ denote the spatial inverse of h , which exists by the spatial monotonicity of ψ and the relation $h_x(x, t) = -\psi_y(e^{-x+\frac{1}{2}A_t}, t)e^{-x+\frac{1}{2}A_t} > 0$, $(x, t) \in \mathbb{R} \times [0, T]$. Observe that by (79) we have $h(-\log(k), 0) = \psi(k, 0) = \mathbb{E}[Z_T I_T(kZ_T)] = x_0$, and hence the underlying Lagrange multiplier satisfies

$$k = e^{-h^{(-1)}(x_0, 0)}. \quad (82)$$

For $t \in [0, T]$, we then have

$$\begin{aligned} X_t^* &= \psi(kZ_t, t) = \psi\left(e^{-h^{(-1)}(x_0, 0)}e^{-M_t-\frac{1}{2}A_t}, t\right) \\ &= \psi\left(e^{-(h^{(-1)}(x_0, 0)+M_t+A_t)+\frac{1}{2}A_t}, t\right) = h(h^{(-1)}(x_0, 0) + M_t + A_t, t), \end{aligned} \quad (83)$$

and (18) follows.

Next, we recall the evolution of the optimal wealth process

$$dX_t^* = \sigma(t)\pi_t^* \cdot (\lambda(t)dt + dW_t), \quad t \in [0, T]. \quad (84)$$

For $t \in [0, T]$, let $N_t := h^{(-1)}(x_0, 0) + M_t + A_t$ and observe that $N_t = h^{(-1)}(X_t^*, t)$, $t \in [0, T]$, by (18). By Itô's formula, the process $X_t^*, t \in [0, T]$, given in (83) satisfies

$$\begin{aligned} dX_t^* &= \left(h_t(N_t, t) + \frac{1}{2}|\lambda(t)|^2 h_{xx}(N_t, t) \right) dt + h_x(N_t, t) dN_t \\ &= h_x\left(h^{(-1)}(X_t^*, t), t\right) \lambda(t) \cdot (\lambda(t)dt + dW_t). \end{aligned} \quad (85)$$

Equating coefficients in (84) and (85), we find that the optimal portfolio process π_t^* is given by

$$\pi_t^* = h_x\left(h^{(-1)}(X_t^*, t), t\right) \sigma^{(-1)}(t) \lambda(t), \quad t \in [0, T],$$

which yields the representation (19) for the optimal portfolio process provided it is admissible. The admissibility is guaranteed by the polynomial growth assumption (16) on I_T and the uniform boundedness of $\lambda(t)$ on $[0, T]$ (see Karatzas and Shreve (1998, Theorem 3.5 (p. 93), and Remark 6.9(ii) (p. 97))). \square

References

Bernard, C., Boyle, P.P. and Vanduffel, S., Explicit Representation of Cost-Efficient Strategies [online]. SSRN, 2012. Available online at: <http://ssrn.com/abstract=1561272> (accessed 13 November 2012).

- Berrier, F., Rogers, L. and Tehranchi, M., A characterization of forward utility functions [online]. Cambridge University, 2009. Available online at: <http://www.statslab.cam.ac.uk/~mike/papers/forward-utilities.pdf> (accessed 13 November 2012).
- Black, F., Individual investment and consumption under uncertainty. In *Portfolio Insurance: A Guide to Dynamic Hedging*, edited by D.L. Luskin. First version 1968, pp. 207–225, 1988 (John Wiley and Sons: New York).
- Cox, A., Hobson, D. and Obłój, J., Utility theory front to back - inferring utility from agents' choices [online]. arXiv.org, 2011. Available online at: <http://arxiv.org/abs/1101.3572> (accessed 13 November 2012).
- Cox, J. and Leland, H., On dynamic investment strategies. *Journal of Economic Dynamics and Control*, 2000, **24**, 1859–1880.
- Dybvig, P., Distributional Analysis of Portfolio Choice. *Journal of Business*, 1988a, **61**, 369–393.
- Dybvig, P., Inefficient Dynamic Portfolio Strategies or How to Throw Away a Million Dollars in the Stock Market. *The Review of Financial Studies*, 1988b, **1**, 67–88.
- Dybvig, P. and Rogers, L., Recovery of preferences from observed portfolio choice in a single realisation. *Review of Financial Studies*, 1997, **10**, 151–174.
- Friedman, A., *Partial Differential Equations of the Parabolic Type* 1964 (Prentice-Hall, Inc.: Englewood, Cliffs, NJ).
- Goldstein, D., Johnson, E. and Sharpe, W., Choosing outcomes versus choosing products: Consumer-focused retirement investment advice. *Journal of Consumer Research*, 2008, **35**, 440–456.
- He, H. and Huang, C-f., Consumption-Portfolio Policies: An Inverse Optimal problem. *Journal of Economic Theory*, 1994, **62**, 257–293.

- Hirschman, I. and Widder, D., *The convolution transform*. 1955 (Princeton University Press: Princeton, NJ).
- Källblad, S., On the classical and forward portfolio choice problems in log-normal markets. Transfer thesis, Oxford University, 2011.
- Karatzas, I. and Shreve, S., *Methods of mathematical finance*, Applications of Mathematics (New York) Vol. 39, 1998 (Springer-Verlag: New York).
- Musiela, M. and Zariphopoulou, T., Optimal asset allocation under forward exponential performance criteria. In *Markov processes and related topics: a Festschrift for Thomas G. Kurtz*, Vol. 4 of *Inst. Math. Stat. Collect.*, pp. 285–300, 2008 (Inst. Math. Statist.: Beachwood, OH).
- Musiela, M. and Zariphopoulou, T., Portfolio choice under dynamic investment performance criteria. *Quantitative Finance*, 2009, **9**, 161–170.
- Musiela, M. and Zariphopoulou, T., Portfolio choice under space-time monotone performance criteria. *SIAM Journal on Financial Mathematics*, 2010, **1**, 326–365.
- Sharpe, W., Individual Risk and Return Preferences: A Preliminary Survey [online]. Stanford University, 2001. Available online at: <http://www.stanford.edu/~wfsarpe/art/rrsurvey/vienna2001.htm> (accessed 13 November 2012).
- Sharpe, W., Goldstein, D. and Blythe, P., The Distribution Builder: A Tool for Inferring Investor Preferences [online]. Stanford University, 2000. Available online at: <http://www.stanford.edu/~wfsarpe/art/qpaper/qpaper.html> (accessed 13 November 2012).
- Widder, D., *The heat equation*. 1975 (Academic Press: New York), Pure and Applied Mathematics, Vol. 67.
- Wilcox, C.H., Positive Temperatures with Prescribed Initial Heat Distributions. *The American Mathematical Monthly*, 1980, **87**, 183–186.